

GROUP THEORY 2024 - 25, SOLUTION SHEET 1

Exercise 1. The Permutation Group S_n

- (1) (a) (135)(24)
 (b) (12356)
 (c) (62)(5431)

(2) Let $x \in \{1, \dots, n\}$ and consider the cycle $(x, a(x), a^2(x), \dots, a^{l-1}(x))$. As $a^p = 1$, we must have $l|p$, but p is prime, so we deduce $l = 1$ or $l = p$. Thus, any $x \in \{1, \dots, n\}$ is either fixed by a or contained in a cycle of length p , which proves the claim.

(3) (a) It is easy to see that a cycle $(x_1 x_2 \dots x_l)$ can be written as a product of transpositions in the following way:

$$(x_1 x_2 \dots x_l) = (x_1 x_l)(x_1 x_{l-1}) \dots (x_1 x_2)$$

We conclude by the fact that any permutation can be written as a product of cycles.

(b) *Lemma:* If $\sigma_1, \dots, \sigma_r$ are transpositions such that $1 = \sigma_1 \dots \sigma_r$, then r is even. By observing that

$$\tau_1 \cdot \tau_2 \cdot \dots \cdot \tau_k = \pi_1 \cdot \pi_2 \cdot \dots \cdot \pi_l$$

\iff

$$1 = \tau_1 \cdot \tau_2 \cdot \dots \cdot \tau_k \cdot \pi_l \cdot \pi_{l-1} \cdot \dots \cdot \pi_1$$

the above lemma lets us conclude.

Proof of the lemma: Clearly, $r \neq 1$. If $r = 2$, we are done. Suppose that $r > 2$ and proceed by induction. Since $(ij) = (ji)$, $\sigma_{r-1} \cdot \sigma_r$ can be expressed in one of the following forms on the right:

$$\begin{aligned} 1 &= (ab)(ab) \\ (ab)(bc) &= (ac)(ab) \\ (ac)(cb) &= (bc)(ab) \\ (ab)(cd) &= (cd)(ab) \end{aligned}$$

If the first case occurs, delete $\sigma_{r-1} \cdot \sigma_r$ from the original product. In the other three cases, we replace the form on the right by its counterpart on the left to obtain a new product of r transpositions that still gives us the identity, but where the rightmost occurrence of the integer a is in the second-from-the-rightmost 2-cycle of the product instead of the rightmost 2-cycle. We repeat the process just described

with the product $\sigma_{r-2} \cdot \sigma_{r-1}$. As before, we obtain either a product of $(r-2)$ 2-cycles equal to the identity or a new product of r 2-cycles, where the rightmost occurrence of a is in the third 2-cycle from the right. Continuing this process, we must obtain a product of $(r-2)$ 2-cycles equal to the identity, because otherwise we have a product equal to the identity in which the only occurrence of the integer a is in the leftmost 2-cycle, and such a product does not fix a , whereas the identity does. By induction, $(r-2)$ is even, so r is also even.

- (c) Both directions of the statement follow easily from the expression of a cycle as product of transpositions presented in (a) and from the invariance of the number of terms of this writing modulo 2, which was proved in (b).

Exercise 2. *The Dihedral Group D_{2n}*

- (1) Recall that D_{2n} is given by the set of bijections of $[n]$ of the form $f_k(x) = x + k \pmod n$, called rotations, or of the form $g_k(x) = -x + k \pmod n$, called reflections. You can check that composing those or taking inverses always yields a rotation or a reflection. Hence it is a group, and since D_{2n} is included in S_n with the same composition law, it is indeed a subgroup.
- (2) We showed in the previous point that $D_6 \leq S_3$, and since both sides have the same cardinality, the inclusion must be an isomorphism.
- (3) Let r be the rotation $x \mapsto x + 1 \pmod 3$ and s be the reflection $x \mapsto -x \pmod 3$. Then for any $f \in D_6$, which is of the form $x \mapsto x + k$ (resp. $x \mapsto -x + k \pmod 3$) for some $k \in \{0, 1, 2\}$, Then $f = r^k$ (resp. $f = r^k s$).
- (4) Let r be the rotation $x \mapsto x + 1 \pmod n$ and s be the reflection $x \mapsto -x \pmod n$.
- (5) You can check that $sr = r^{-1}s \neq rs$.

You should verify visually that s and r generates D_6 and that they do not commute.

Exercise 3. *Some more examples of groups.*

- (1) G is cyclic generated by i of order 4. Hence it is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
- (2) The elements of complex circle K can be written as

$$K = \{e^{2\pi ik} \mid k \in [0, 1)\} = \{e^{2\pi ik} \mid k \in \mathbb{R}\}.$$

With this representation, the complex multiplication is $e^{2\pi ik} e^{2\pi il} = e^{2\pi i(k+l)}$. The unit is $1 = e^{2\pi i0} = e^0$ and the inverse of $e^{2\pi ik}$ is $e^{-2\pi ik}$.

- (3) If $x \in \mathbb{C}$ is such that $x^n - 1 = 0$, then $|x|^n = 1$ which implies that the norm of x is $|x| = 1$. Hence the elements of H are all contained in the complex circle K , i.e. $H \subseteq K$.

These elements are precisely

$$H = \{e^{\frac{2\pi ik}{n}} \mid k \in \{0, \dots, n-1\}\}. \quad (1)$$

You can check that these are all distinct n -th roots of unity. Since $x^n - 1$ is a polynomial of degree n , it can't have more than n roots, which proves the equality (1). Moreover we readily check that $H \subseteq K$ is a subgroup, i.e. that H is stable under multiplication and taking inverses.

Exercise 4. Group Actions

(1) We show that $\text{Stab}_G(x) \subseteq G$ is a subgroup of G . For all $g, h \in \text{Stab}_G(x)$ and $x \in X$:

- (a) $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$, so $gh \in \text{Stab}_G(x)$;
- (b) $g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = x$. so $g^{-1} \in \text{Stab}_G(x)$

(2) Fix $k \in X = \{1, \dots, n\}$. Its stabilizer group is the set of permutation that fixes k , which we observe is isomorphic to S_{n-1} . In formula this is:

$$\text{Stab}_G(k) = \{\sigma \in S_n \mid \sigma(k) = k\} \cong \{\tau \in S_{n-1}\} \cong S_{n-1}.$$

Note that for *different* $k \in X$, the subgroups $\text{Stab}_G(k) \cong S_{n-1} \subset S_n$ are realized by *different* injective homomorphisms $S_{n-1} \rightarrow S_n$.

(3) A direct computation shows that for $x = E_{ij}$, its stabilizer subgroup is given by the set of invertible matrices $A \in GL_n(K)$ such that

$$A_{ki} = \begin{cases} 0 & \text{if } k \neq i; \\ 1 & \text{if } k = i. \end{cases}$$

For $x = B$ an invertible matrix, $AB = B$ implies that $A = I_n$ is the identity matrix, so that $\text{Stab}_G(B)$ is the trivial group.

(4) The stabilizer group at $M \in M_n(K)$ is the set of invertible matrices $A \in GL_n(K)$ such that $AMA^{-1} = M$, that is those A such that $AM = MA$. Since every matrix commutes with a diagonal matrix $M = \lambda I_n$ we have

$$\text{Stab}_G(\lambda I_n) = GL_n(K).$$

If M is a diagonal matrix with *distinct* entries, a direct computation shows that its stabilizer group is the group of invertible diagonal matrices.

(5) We find that

$$\text{Stab}_G(x) = 1 \quad \text{and} \quad \text{Stab}_{\bullet G}(x) = \{g \in G \mid gx = xg\} = Z_G(x).$$

The latter group is called the *centralizer* of G at $x \in G$. Let us compute it for the different groups.

Without loss of generality, suppose that $\sigma = (ij) = (12)$. The centralizer of (12) consists of all permutations that either leave 1 and 2 fixed or swap them while permuting the rest of the elements freely. Hence, the centralizer of (12) in S_n is

$$C_{S_n}((12)) \cong S_2 \times S_{n-2}$$

where S_2 permutes 1 and 2, and S_{n-2} permutes the remaining $n - 2$ elements.

Without loss of generality suppose that $\tau = (123)$. Suppose that $\sigma \in S_n$ is such that $\sigma(k) = l \notin \{1, 2, 3\}$ for some $k \in \{1, 2, 3\}$. Then $\sigma(123)\sigma^{-1}(l) \neq l = (123)(l)$ which shows that $\sigma \notin C_{S_n}((123))$. Hence we find that the centralizer of (123) consists of all permutations σ such that $\sigma|_{[3]} : [3] \rightarrow [3]$ belong to $C_{S_3}((123))$ while it permuting the rest of the elements $\{4, \dots, n\}$ freely. It is clear that the centralizer of (123) in S_3 are the powers of $\tau = (123)$, i.e. the subgroup $\langle \tau \rangle = \{1_{S_3}, \tau, \tau^2\} \subset S_3$. Hence the centralizer of (123) in S_n is

$$C_{S_n}((123)) \cong C_{S_3}((123)) \times S_{n-3} \cong \mathbb{Z}/3\mathbb{Z} \times S_{n-3}$$

where $\mathbb{Z}/3\mathbb{Z} \cong \langle \tau \rangle$ and S_{n-3} permutes the rest of the elements.

Since $\mathbb{Z}/n\mathbb{Z}$ is abelian, its centralizer at all elements is the group itself.

(6) We find that $\text{Stab}_G(0) = C_2 \cong \mathbb{Z}/2\mathbb{Z}$ and that $\text{Stab}_G(x) = 1$ for all $x \neq 0$.

Exercise 5. Some Geometric Actions

(1) First we show that the order of G is 24. Fix a vertex v of the cube and an edge e which is adjacent to the vertex v . The key observation is that a rotation of the cube is determined by the choice of sending v to some vertex w and e to an edge which is adjacent to w . Since there are 8 vertices on a cube and 3 edges adjacent to each vertex, we obtain that $|G| = 8 \times 3 = 24$.

Now we show that the group homomorphism $\phi : G \rightarrow S_4$ induced by the action of G on the long diagonals is injective. Let $r \in G$ be a rotation such that r fixes all the long diagonals of the cube. One observes using a case by case analysis that this is possible only if r is the identity rotation. Unwinding the definition of ϕ , this precisely means that ϕ is injective.

Since an injective map between sets of the same cardinality is also surjective, we obtain that ϕ is an isomorphism. \square

(2) First one observes that D_{12} does indeed act on S which induces a group homomorphism $\psi : D_{12} \rightarrow S_5$. Note that the only elements of D_{12} whose action fixes all the long diagonals are the identity and rotation by π radians. But rotation by π radians switches the two triangles T_1 and T_2 . Hence the only element of D_{12} which fixes all elements of S is the identity element. This implies that ψ is injective. \square